

GAUSSIAN INTEGRABILITY OF DISTANCE FUNCTION UNDER THE LYAPUNOV CONDITION

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ABSTRACT. In this note we give a direct proof of the Gaussian integrability of distance function as $\mu e^{\delta d^2(x, x_0)} < \infty$ for some $\delta > 0$ provided the Lyapunov condition holds for symmetric diffusion Markov operators, which answers a question proposed in Cattiaux-Guillin-Wu [6, Page 295]. The similar argument still works for diffusions processes with unbounded diffusion coefficients and for jump processes such as birth-death chains. An analogous discussion is also made under the Gozlan's condition arising from [9, Proposition 3.5].

1. INTRODUCTION

In this note, we will investigate how to directly derive the Gaussian integrability from two kinds of criteria for the Talagrand's inequality W_2H , say the Lyapunov condition and Gozlan's condition presented in a symmetric diffusion Markov setting. Referring to Bakry-Gentil-Ledoux [2], in the sequel we denote by E a complete connected Riemannian manifold of finite dimension, d the geodesic distance, dx the volume measure, $\mu(dx) = e^{-V(x)}dx$ a probability measure with $V \in C^2(E)$, $L = \Delta - \nabla V \cdot \nabla$ the μ -symmetric diffusion operator, $\Gamma(f, g) = \nabla f \cdot \nabla g$ the carré du champ operator, and \mathcal{E} the associated Dirichlet form, which satisfy the formula for integration by parts

$$\int \nabla f \cdot \nabla g d\mu = - \int f Lg d\mu, \quad f \in \mathcal{D}(\mathcal{E}), g \in \mathcal{D}(L).$$

First of all, say $W \geq 1$ is a Lyapunov function if there exist two constants $b \geq 0$ and $c > 0$ such that for some $x_0 \in E$ and any $x \in E$

$$(1.1) \quad LW \leq (-cd^2(x, x_0) + b)W.$$

More generally, to avoid assuming the integrability and second-order differentiability of W , it is convenient to introduce a locally Lipschitz function $U > 0$ such that in the sense of distribution

$$(1.2) \quad LU + |\nabla U|^2 \leq -cd^2(x, x_0) + b,$$

which means that for any nonnegative $h \in C_c^\infty(E)$ holds

$$\int (LU + |\nabla U|^2) h d\mu := \int ULh + |\nabla U|^2 h d\mu \leq \int (-cd^2(x, x_0) + b) h d\mu.$$

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When $W \in C^2(E)$, (1.1) and (1.2) are equivalent by taking $U = \log W$. And it is not hard to see that (1.1) implies a weaker version for some c' , b' and R

$$(1.3) \quad LW \leq -c'W + b'\mathbf{1}_{B(0,R)}.$$

The Lyapunov condition plays a powerful role in studying coercive functional inequalities or estimating convergence rate of Markov processes, which even works as a substitute of curvature-dimension condition in some cases. Cattiaux-Guillin [4] gave a comprehensive review on this topic, and here we would like to take partial literature into account. A simple proof of the Poincaré inequality through (1.3) can be found in Bakry-Barthe-Cattiaux-Guillin [1]. The L^1 transport-information inequality W_1I was discussed further under (1.1) by Guillin-Léonard-Wu-Yao [12]. Then Cattiaux-Guillin-Wu [6] showed the Talagrand's inequality and logarithmic Sobolev inequality (LSI for short) provided (1.2), which was also applied to weighted LSIs for heavy tailed distributions by [7]. Most recently, Guillin-Joulin [10] obtained non-Gaussian concentration estimates by means of functional inequalities with some kind of Lyapunov condition yet.

According to [6, Lemma 3.5], it was proved that if (1.2) holds, there exist some $\delta > 0$ and $x_0 \in E$ such that

$$(1.4) \quad \int e^{\delta d^2(x, x_0)} d\mu(x) < \infty,$$

which is necessary to derive W_2H . Their proof starts from (1.2) and the spectral gap to show W_1I due to [12]. It then follows a L^1 transport-entropy inequality W_1H by Guillin-Léonard-Wang-Wu [11], which is equivalent to (1.4) by Djellout-Guillin-Wu [8]. The strategy relies on a series of works on transport inequalities, thereupon the authors of [6] feel interested in finding a simple or direct proof of (1.4), see [6, Page 295].

Indeed, there exists an elementary proof, and we actually obtain

Proposition 1.1. *If (1.2) holds, then $\mu e^{\delta d^2(x, x_0)} < \infty$ for any $\delta < \sqrt{c}$.*

Remark 1.2. *The upper bound for δ is sharp. For instance, let $d\mu = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}|x|^2} dx$ and $L = \frac{d^2}{dx^2} - x \frac{d}{dx}$ associated to one-dimensional Ornstein-Uhlenbeck process, then $W = e^{\frac{1}{4}|x|^2}$ satisfies $LW \leq (-\frac{1}{4}|x|^2 + \frac{1}{2})W$, which exactly gives $\delta < \sqrt{c} = \frac{1}{2}$.*

Remark 1.3. *A weak version $LW \leq (-cd^p(x, x_0) + b)W$ with $p < 2$ is not enough to derive the Gaussian integrability, since $W = \exp(\frac{1}{2}(1 + |x|^2)^q)$ with $2(q-1) = p$ fulfills (1.2) for $d\mu = \frac{1}{Z} e^{-(1+|x|^2)^{\frac{p}{2}}} dx$, where Z is a normalization factor.*

The same argument can be extended to diffusion processes with unbounded diffusion coefficients. Define an infinitesimal generator in \mathbb{R}^m

$$L_a = \frac{1}{2} \sum_{i,j=1}^m a^{ij}(x) \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} + \sum_{i=1}^m b^i(x) \frac{\partial}{\partial x_i},$$

where $A = (a^{ij})_{i,j=1}^m$ is symmetric positive-definite and $b^i = \frac{1}{2} \left(\sum_{j=1}^m \frac{\partial a^{ij}}{\partial x_j} - a^{ij} \frac{\partial V}{\partial x_j} \right)$ so that L_a admits an invariant probability measure $d\mu(x) = e^{-V} dx$. Then define the Carrédu champ operator by means of

$$\Gamma_a(f, g) = \frac{1}{2} [L_a(fg) - fL_ag - gL_af] = \frac{1}{2} \langle \nabla f, A \nabla g \rangle,$$

which satisfies the integration by parts formula for $f, g \in C_c^\infty(\mathbb{R}^m)$

$$-\int f L_a g d\mu = \int \Gamma_a(f, g) d\mu =: \mathcal{E}_a(f, g).$$

Thanks to the Lyapunov type criterion by Stroock-Varadhan [13, Theorem 10.2.1], it can be quickly derived that L_a corresponds to a non-explosive diffusion process provided that (1.1) holds by substituting L to L_a

$$(1.5) \quad L_a W \leq (-cd^2(x, x_0) + b)W$$

with $\lim_{|x| \rightarrow \infty} W = \infty$.

However, if a^{ij} is unbounded, (1.5) is not enough to get the Gaussian integrability for μ . Consider one-dimensional case, when $a^{ij} = a(x) = o(|x|^4)$, we take $V = \frac{x^2}{2\sqrt{a}}$ and $W = e^{\delta V}$ for small $\delta > 0$ so that (1.5) holds but V has a growth rate slower than quadratic. On the other hand, when $a(x) = O(|x|^4)$ or grows even faster, (1.5) is useless to yield a Poincaré type inequality so that we have no effective calculus on the integrability of $e^{\delta d^2(x, x_0)}$. For this reason, a stronger condition is necessary.

Proposition 1.4. *Let λ_{\max} be the maximal eigenvalue of A satisfying $\mu\lambda_{\max} < \infty$. Suppose there exists a Lyapunov function $W \geq 1$ with two constants $b \geq 0$ and $c > 0$ such that for some $x_0 \in \mathbb{R}^m$ and any $x \in \mathbb{R}^m$*

$$(1.6) \quad L_a W \leq (-cd^2(x, x_0) + b)\lambda_{\max} W.$$

Then $\mu(e^{\delta d^2(x, x_0)} \lambda_{\max}) < \infty$ for any $\delta < \sqrt{c}$.

Remark 1.5. (1.6) is natural, for instance, if there exist V and W satisfying (1.1) over \mathbb{R} , then (1.6) follows automatically provided that $\lim_{|x| \rightarrow \infty} \frac{a'W'}{aW|x|^2} = 0$. Moreover, there is no need to assume $\lambda_{\max} \geq \lambda > 0$ uniformly on \mathbb{R}^m .

Another possible extension is about jump processes (see Bass [3]). To clarify the effect from jumps part, we simply consider the infinitesimal generator of the form

$$L_\nu = \int_{\mathbb{R}^m - \{0\}} [f(x+y) - f(x) - \nabla f \cdot y \mathbf{1}_{0 < |y| < 1}(y)] \nu(x, dy),$$

Where ν satisfies $\int_{\mathbb{R}^m - \{0\}} \min\{1, |y|^2\} \nu(x, dy) < \infty$. Suppose that L_ν admits an invariant probability measure μ , and the Carrédu champ operator

$$\begin{aligned} \Gamma_\nu(f, g) &= \frac{1}{2} [L_\nu(fg) - fL_\nu g - gL_\nu f] \\ &= \frac{1}{2} \int_{\mathbb{R}^m - \{0\}} [f(x+y) - f(x)][g(x+y) - g(x)] \nu(x, dy) \end{aligned}$$

fulfills the integration by parts formula for $f, g \in C_c^\infty(\mathbb{R}^m)$

$$-\int f L_\nu g d\mu = \int \Gamma_\nu(f, g) d\mu =: \mathcal{E}_\nu(f, g).$$

Then define an intrinsic (pseudo)metric according to Sturm [14, Definition 6.5]

$$\rho(x, y) := \sup\{f(x) - f(y) : \Gamma_\nu(f, f) \leq 1\},$$

which gives $\Gamma_\nu(\rho(x, x_0), \rho(x, x_0)) \leq 1$ if $\rho(x, x_0) \in \mathcal{D}(\mathcal{E}_\nu)$. For convenience, we also require that $\lim_{|x| \rightarrow \infty} \rho(x, x_0) = \infty$.

The setting includes discrete Markov chains. For example, consider a birth-death process on \mathbb{N} with strictly positive birth rates b_i and death rates d_i except $d_0 = 0$.

Let $r_0 = 1$ and $r_i = \frac{b_0 b_1 \cdots b_{i-1}}{d_1 d_2 \cdots d_i}$ for $i \geq 1$, we can take $\nu(i, y) = b_i \delta_1(y) + d_i \delta_{-1}(y)$ and $\mu(i) = \frac{r_i}{r_0 + r_1 + \cdots}$ provided the series converges, and then \mathcal{E}_ν has an alternative expression $\mathcal{E}_\nu(f, g) = \sum_{i=0}^{\infty} [f(i+1) - f(i)][g(i+1) - g(i)] b_i \mu_i$, which determines the intrinsic metric $\rho(i, j) = b_i^{-\frac{1}{2}} + b_{i+1}^{-\frac{1}{2}} + \cdots + b_{j-1}^{-\frac{1}{2}}$ for $i \leq j$.

Proposition 1.6. *Suppose there exist some $x_0 \in \mathbb{R}^m$ and a constant $K > 0$ such that for all $x \in \mathbb{R}^m$ and all $y \in \text{Supp } \nu$*

$$(1.7) \quad |\rho^2(x+y, x_0) - \rho^2(x, x_0)| \leq K.$$

Suppose also there exists a Lyapunov function $W \geq 1$ with two constants $b \geq 0$ and $c > 0$ such that for any $x \in \mathbb{R}^m$

$$(1.8) \quad L_\nu W \leq (-c\rho^2(x, x_0) + b)W.$$

Then $\mu e^{\delta \rho^2(x, x_0)} < \infty$ for $\delta < C \min\{\sqrt{c}, K^{-1}\}$ with some multiple $C \in (0, 1]$.

Remark 1.7. *For a birth-death process referring to Cattiaux-Guillin-Wang-Wu [5], let $b_i = d_i = i^a \log^\alpha(i+1)$ with $a \geq 2$ and $\alpha \in \mathbb{R}$ except $b_0 = 1$, let $W = 1 + i^\gamma$ with $0 < \gamma < 1$, then $\mu(i) \asymp b_i^{-1}$, $L_\nu W \leq -c i^{a-2} \log^\alpha(i+1)W$. Take $a = 2, \alpha = 1, \gamma = \frac{1}{2}$, it follows $\rho(i, 0) \asymp \log^{\frac{1}{2}}(i+1)$ satisfying (1.7-1.8) and then $\mu e^{\delta \rho^2} < \infty$ for $\delta < 1$. If $a = 2, \alpha < 1, \gamma = \frac{1}{2}$, (1.7) holds, but (1.8) fails and so does the Gaussian integrability; if $b_i = (i+1)^{\frac{1}{2}}$ and $d_i = i b_i$, then $\mu(i) \asymp (i b_i)^{-1}$, $\rho(i, 0) \asymp i^{\frac{3}{4}}$ and (1.8) holds for $W = 2^i$, but (1.7) fails and so does the Gaussian integrability again.*

We further investigate another criterion for transport-entropy inequalities. According to Gozlan [9, Proposition 3.5], let μ be a probability on \mathbb{R}^m , suppose there exists $\omega \in C^3(\mathbb{R})$ with $\omega'(0) > 0$, $\left| \frac{\omega^{(3)}}{\omega^3} \right| \leq M$ for some constant M , and

$$(1.9) \quad \liminf_{|x| \rightarrow \infty} \frac{1}{u^2} \sum_{i=1}^m \left[\frac{1}{10} \left(\frac{\partial V}{\partial x_i} \right)^2 \left(\frac{x}{u} \right) - \frac{\partial^2 V}{\partial x_i^2} \left(\frac{x}{u} \right) \right] \frac{1}{\omega'(x_i)^2} > mM$$

for some constant $u > 0$, then a transport-entropy inequality holds with the cost function $d_\omega(x, y) = \left(\sum_{i=1}^m |\omega(x_i) - \omega(y_i)|^2 \right)^{\frac{1}{2}}$. An interesting case is to set

$$\omega(t) = \int_0^t \sqrt{1+s^2} ds = \frac{t}{2} \sqrt{1+t^2} + \frac{1}{2} \log |t + \sqrt{1+t^2}|$$

satisfying $\omega'(0) = 1$ and $\left| \frac{\omega^{(3)}}{\omega^3}(t) \right| = (1+t^2)^{-3} \leq 1$, which corresponds to $W_2 H$.

In [6], it was pointed out that (1.9) is not comparable to the Lyapunov condition (1.2) in general. Using the similar argument, we still have

Proposition 1.8. *If the Gozlan's type condition holds, i.e.*

$$(1.10) \quad \liminf_{|x| \rightarrow \infty} \sum_{i=1}^m \left[\frac{23}{27} \left(\frac{\partial V}{\partial x_i} \right)^2(x) - \frac{\partial^2 V}{\partial x_i^2}(x) \right] \frac{1}{1+x_i^2} \geq m,$$

then $\mu e^{\delta |x|^2} < \infty$ for any $\delta < \frac{2(\sqrt{m}-\sqrt{m-1})}{3\sqrt{3m}}$.

Remark 1.9. *To yield the Gaussian integrability, or equivalently $W_1 H$, the original constant $\frac{1}{10}$ in (1.9) can be increased to arbitrary $a < 1 - \frac{4}{27} \frac{m-1}{m}$. So it is convenient to take $a = \frac{23}{27}$. Except $m = 1$, it is unlikely to allow a approaching 1, according to the estimates in Lemma 3.1 below.*

The next two sections will supply the proofs of all propositions respectively.

2. PROOF OF PROPOSITION 1.1, 1.4 AND 1.6

Under the Lyapunov condition (1.2), [6, Lemma 3.4] asserts

$$(2.1) \quad \int h^2(x) d^2(x, x_0) d\mu(x) \leq \frac{1}{c} \int |\nabla h|^2 d\mu + \frac{b}{c} \int h^2 d\mu, \quad \forall h \in \mathcal{D}(\mathcal{E}).$$

The technique of proof is the same as in [1, Page 64].

Now, we prove Proposition 1.1.

Proof. Let $\beta_n = \int d^{2n}(x, x_0) d\mu$, which satisfies a recursion by using (2.1) that

$$(2.2) \quad \begin{aligned} \beta_n &= \int d^{2(n-1)}(x, x_0) d^2(x, x_0) d\mu \\ &\leq \frac{1}{c} \int |\nabla d^{n-1}(x, x_0)|^2 d\mu + \frac{b}{c} \beta_{n-1} = \frac{(n-1)^2}{c} \beta_{n-2} + \frac{b}{c} \beta_{n-1}. \end{aligned}$$

Since $\beta_0 = 1$ and $\beta_1 \leq \frac{b}{c}$, we get the integrability of all $d^{2n}(x, x_0)$.

Combining the Hölder inequality with (2.2) gives

$$\beta_n = \int d^{n+1}(x, x_0) d^{n-1}(x, x_0) d\mu \leq \beta_{n+1}^{\frac{1}{2}} \beta_{n-1}^{\frac{1}{2}} \leq \left(\frac{n^2}{c} \beta_{n-1} + \frac{b}{c} \beta_n \right)^{\frac{1}{2}} \beta_{n-1}^{\frac{1}{2}},$$

which implies

$$\beta_n \leq \frac{\frac{b}{c} + \sqrt{\frac{b^2}{c^2} + \frac{4n^2}{c}}}{2} \beta_{n-1} \leq \left(\frac{b}{c} + \frac{n}{\sqrt{c}} \right) \beta_{n-1}.$$

Taking any $\gamma > \frac{1}{\sqrt{c}}$ gives $\frac{b}{c} + \frac{n}{\sqrt{c}} \leq \gamma n$ for big n , which yields some $C > 0$ such that

$$\beta_n \leq C \gamma^n n!, \quad \forall n \geq 1.$$

Hence, for any $\delta < \gamma^{-1} < \sqrt{c}$, we have by the Fatou's lemma

$$(2.3) \quad \begin{aligned} \int e^{\delta d^2(x, x_0)} d\mu &= \int \lim_{k \rightarrow \infty} \sum_{n=0}^k (\delta d^2(x, x_0))^n / n! d\mu \\ &\leq \liminf_{k \rightarrow \infty} \int \sum_{n=0}^k (\delta d^2(x, x_0))^n / n! d\mu = \liminf_{k \rightarrow \infty} \sum_{n=0}^k \delta^n \beta_n / n! \leq \frac{C}{1 - \delta \gamma}. \end{aligned}$$

The proof is completed. \square

The proof of Proposition 1.4 is almost the same.

Proof. Using the Lyapunov condition (1.6) with the technique from [1, Page 64] gives a similar inequality for $h \in \mathcal{D}(\mathcal{E}_a)$ as (2.1) that

$$\begin{aligned} \int h^2(x) d^2(x, x_0) \lambda_{\max} d\mu &\leq \frac{1}{c} \int h^2 \cdot \frac{-L_a W}{W} d\mu + \frac{b}{c} \int h^2 \lambda_{\max} d\mu \\ &= \frac{1}{c} \int \Gamma_a\left(\frac{h^2}{W}, W\right) d\mu + \frac{b}{c} \int h^2 \lambda_{\max} d\mu \\ &= \frac{1}{c} \int \Gamma_a(h, h) - W^2 \Gamma_a\left(\frac{h}{W}, \frac{h}{W}\right) d\mu + \frac{b}{c} \int h^2 \lambda_{\max} d\mu \\ &\leq \frac{1}{c} \int |\nabla h|^2 \lambda_{\max} d\mu + \frac{b}{c} \int h^2 \lambda_{\max} d\mu. \end{aligned}$$

Let $\beta_n = \int d^{2n}(x, x_0) \lambda_{\max} d\mu$, which satisfies

$$\begin{aligned} \beta_n &= \int d^{2(n-1)}(x, x_0) d^2(x, x_0) \lambda_{\max} d\mu \\ &\leq \frac{1}{c} \int |\nabla d^{n-1}(x, x_0)|^2 \lambda_{\max} d\mu + \frac{b}{c} \beta_{n-1} = \frac{(n-1)^2}{c} \beta_{n-2} + \frac{b}{c} \beta_{n-1}. \end{aligned}$$

Following rest steps in the previous proof, we get the Gaussian integrability. \square

At the end of this section, we prove Proposition 1.6 by a little different method.

Proof. The strategy contains three steps.

Step 1. Denote $\rho_t(x) = \sqrt{\rho^2(x, x_0) + t}$ with some parameter $t > 0$. Using the technique in [1, Page 64] again, we have by Condition (1.8) that for $h \in \mathcal{D}(\mathcal{E}_\nu)$

$$\begin{aligned} \int h^2 \rho_t^2 d\mu &\leq \frac{1}{c} \int h^2 \cdot \frac{-L_a W}{W} d\mu + \left(\frac{b}{c} + t\right) \int h^2 d\mu \\ &= \frac{1}{c} \int \Gamma_\nu\left(\frac{h^2}{W}, W\right) d\mu + \frac{b+ct}{c} \int h^2 d\mu \\ &= \frac{1}{c} \cdot \frac{1}{2} \int \int_{\mathbb{R}^m - \{0\}} \left| h(x+y) \frac{W(x)^{\frac{1}{2}}}{W(x+y)^{\frac{1}{2}}} - h(x) \frac{W(x+y)^{\frac{1}{2}}}{W(x)^{\frac{1}{2}}} \right|^2 \\ &\quad + |h(x+y) - h(x)|^2 \nu(x, dy) \mu(dx) + \frac{b+ct}{c} \int h^2 d\mu \\ &\leq \frac{1}{c} \int \Gamma_\nu(h, h) d\mu + \frac{b+ct}{c} \int h^2 d\mu. \end{aligned}$$

Step 2. Basically, our aim is to estimate $\int_\Omega e^{\delta \rho(x, x_0)^2} d\mu(x)$ for any bounded domain Ω , while the integration by parts requires to regularize the characteristic function $\mathbf{1}_\Omega$. It is usually a routine but with a few tricks in this case.

Define a family of $\phi_r \in C^1(\mathbb{R}^+)$ with any $r > 0$ and some constant $N > 0$ as

$$\phi_r(s) = \begin{cases} 1, & s \leq r; \\ 2\left(\frac{s-r}{N}\right)^3 - 3\left(\frac{s-r}{N}\right)^2 + 1, & r < s < r+N; \\ 0, & s \geq r+N, \end{cases}$$

which satisfies $0 \leq \phi_r \leq 1$ and $|\phi'_r| \leq \frac{3}{2N} \mathbf{1}_{r < s < r+N}$.

Let $f = e^{\frac{\delta}{2} \rho_t^2}$ and $f_r = \phi_r(\rho_t^2) f$. Let $h_r = \frac{f_r}{\rho_t}$, we have by Step 1

$$(2.4) \quad \int f_r^2 d\mu = \int h_r^2 \rho_t^2 d\mu \leq \frac{1}{c} \int \Gamma_\nu(h_r, h_r) d\mu + \frac{b+ct}{c} \int h_r^2 d\mu.$$

For convenience, rewrite $h_r = \phi_r(\rho_t^2) \psi(\rho_t)$ by putting $\psi(s) := \frac{e^{\frac{\delta}{2} s^2}}{s}$.

Take $t = 2\delta^{-1}$ so that ψ is increasing on $[\sqrt{t}, \infty)$. Using the mean value theorem respectively to ψ and ϕ_r yields that for any $x \in \mathbb{R}^m$ and $y \in \text{Supp} \nu$, there exist ξ and ζ both falling between $\rho(x+y)$ and $\rho(x)$ such that

$$\begin{aligned} &|h_r(x+y) - h_r(x)| \\ &\leq \phi_r(\rho_t^2(x)) \cdot |\psi(\rho_t(x+y)) - \psi(\rho_t(x))| \\ &\quad + \psi(\rho_t(x+y)) \cdot |\phi_r(\rho_t^2(x+y)) - \phi_r(\rho_t^2(x))| \\ &= \phi_r(\rho_t^2(x)) \cdot |\delta - \xi^{-2}| e^{\frac{\delta}{2} \xi^2} \cdot |\rho_t(x+y) - \rho_t(x)| \\ &\quad + \psi(\rho_t(x+y)) \cdot |2\zeta \phi'_r(\zeta^2)| \cdot |\rho_t(x+y) - \rho_t(x)|, \end{aligned}$$

which implies by Condition (1.7) that

$$\begin{aligned} |h_r(x+y) - h_r(x)| &\leq \frac{\delta}{2} e^{\frac{\delta}{2}K} \cdot f_r(x) \cdot |\rho_t(x+y) - \rho_t(x)| \\ &\quad + \frac{3e^{\frac{\delta}{2}K}}{N} \cdot f(x) \mathbf{1}_{r-K < \rho_t^2(x) < r+N+K} \cdot |\rho_t(x+y) - \rho_t(x)|. \end{aligned}$$

Due to $\Gamma_\nu(\rho_t, \rho_t) \leq 1$, it follows

$$\begin{aligned} \Gamma_\nu(h_r, h_r) &= \frac{1}{2} \int_{\mathbb{R}^m - \{0\}} |h_r(x+y) - h_r(x)|^2 \nu(x, dy) \\ &\leq \frac{1}{2} \delta^2 e^{\delta K} f_r^2(x) + \frac{18e^{\delta K}}{N^2} f^2(x) \mathbf{1}_{r-K < \rho_t^2(x) < r+N+K} \\ &\leq \frac{1}{2} \delta^2 e^{\delta K} f_r^2(x) + \frac{18e^{\delta(2N+3K)}}{N^2} e^{\delta(r-N-K)} \mathbf{1}_{r-N-K < \rho_t^2(x) < r+N+K}. \end{aligned}$$

Let $\eta_1 = \frac{\delta^2}{2c} e^{\delta K}$ and $\eta_2 = \frac{18e^{\delta(2N+3K)}}{N^2 c}$, inserting the above estimate into (2.4) gives

$$\begin{aligned} \int f_r^2 d\mu &\leq \eta_1 \int f_r^2 d\mu + \frac{b+ct}{c} \int h_r^2 d\mu \\ &\quad + \eta_2 e^{\delta(r-N-K)} \mu\{r-N-K < \rho_t^2 < r+N+K\}. \end{aligned}$$

Step 3. Choose some big N and small δ so that $\eta_1 + 2\eta_2 < 1$. Since μ is a probability, there exists a sequence of $n_k \in \mathbb{N}$ such that for each $r_k = n_k(N+K)$

$$\mu\{r_k - N - K < \rho_t^2 < r_k\} \geq \mu\{r_k < \rho_t^2 < r_k + N + K\},$$

which implies

$$\begin{aligned} &e^{\delta(r-N-K)} \mu\{r_k - N - K < \rho_t^2 < r_k + N + K\} \\ &\leq 2 \int f^2 \mathbf{1}_{r_k - N - K < \rho_t^2 \leq r_k} d\mu \leq 2 \int f_{r_k}^2 d\mu. \end{aligned}$$

It follows from Step 2

$$\int f_{r_k}^2 d\mu \leq (\eta_1 + 2\eta_2) \int f_{r_k}^2 d\mu + \frac{b+ct}{c} \int h_{r_k}^2 d\mu,$$

and thus

$$\int f_{r_k}^2 d\mu \leq \frac{b+ct}{c(1-\eta_1-2\eta_2)} \int h_{r_k}^2 d\mu =: C \int h_{r_k}^2 d\mu.$$

Recall $h_r = \frac{f_r}{\rho_t}$, fix a domain Ω with $\rho_t^2 \geq 2C$ on Ω^c , which means for $r_k > \text{diam}\Omega$

$$\int f_{r_k}^2 d\mu \leq C \int_{\Omega} \frac{f^2}{\rho_t^2} d\mu + \frac{1}{2} \int_{\Omega^c} f_{r_k}^2 d\mu.$$

Consequently, we get $\int f^2 d\mu = \lim_{k \rightarrow \infty} \int f_{r_k}^2 d\mu \leq 2C \int_{\Omega} \frac{f^2}{\rho_t^2} d\mu < \infty$. \square

3. PROOF OF PROPOSITION 1.8

We firstly derive a Poincaré like inequality.

Lemma 3.1. *If the Gozlan's type condition (1.10) holds, there exist two constants λ_1 and λ_2 with big R such that for any $h \in \mathcal{D}(\mathcal{E})$*

$$\int h^2 d\mu \leq \lambda_1 \int \sum_{i=1}^m \frac{|h'_i|^2}{1+x_i^2} d\mu + \lambda_2 \int_{B(0,R+1)} h^2 d\mu.$$

Proof. For convenience, denote $a = \frac{23}{27}$, $d\nu_i = e^{-aV} dx_i$ and

$$d\hat{x}_i = dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_m$$

so that $d\mu = e^{-(1-a)V} d\nu_i d\hat{x}_i$. Define $\phi_r \in C^1(\mathbb{R}^n)$ as

$$\phi_r(x) = \begin{cases} 1, & |x| \leq r; \\ 2(|x| - r)^3 - 3(|x| - r)^2 + 1, & r < |x| < r+1; \\ 0, & |x| \geq r+1, \end{cases}$$

which satisfies $0 \leq \phi_r \leq 1$ and $|(\phi_r)'_i| \leq 6 \frac{|x_i|}{|x|} \sqrt{1 - \phi_r}$. The proof has three steps.

Step 1. For any $\varepsilon > 0$, there exists $R > 0$ by (1.10) such that for all $|x| \geq R$

$$\sum_{i=1}^m (a|V'_i|^2 - V''_{ii}) \frac{1}{1+x_i^2} \geq m - \varepsilon.$$

It follows for any $h \in \mathcal{D}(\mathcal{E})$

$$\begin{aligned} (m - \varepsilon) \int h^2 d\mu &= (m - \varepsilon) \int h^2 \phi_R + h^2 (1 - \phi_R) d\mu \\ &\leq (m - \varepsilon) \int h^2 \phi_R d\mu + \int h^2 (1 - \phi_R) \sum_{i=1}^m (a|V'_i|^2 - V''_{ii}) \frac{1}{1+x_i^2} d\mu \\ (3.1) &= (m - \varepsilon) \int h^2 \phi_R d\mu + \sum_{i=1}^m \int \frac{h^2 (1 - \phi_R) e^{-(1-a)V}}{(1+x_i^2)} (a|V'_i|^2 - V''_{ii}) d\nu_i d\hat{x}_i. \end{aligned}$$

Set $U^{(i)} = \frac{h^2 (1 - \phi_R) e^{-(1-a)V}}{1+x_i^2}$. For the reader's convenience, recall the integration by parts formula satisfied by ν_i , we have

$$\begin{aligned} \int U^{(i)} (a|V'_i|^2 - V''_{ii}) d\nu_i d\hat{x}_i &= \int (U^{(i)})'_i V'_i d\nu_i d\hat{x}_i \\ &= \int \left[2h h'_i V'_i (1 - \phi_R) - (\phi_R)'_i h^2 V'_i - \frac{2x_i}{1+x_i^2} h^2 V'_i (1 - \phi_R) \right. \\ &\quad \left. - (1-a)h^2 |V'_i|^2 (1 - \phi_R) \right] \frac{1}{1+x_i^2} d\mu. \end{aligned}$$

Using the Cauchy-Schwarz inequality gives for any positive $\varepsilon_1, \varepsilon_2$ and ε_3

$$\begin{aligned} 2h h'_i V'_i &\leq \varepsilon_1 h^2 |V'_i|^2 + \varepsilon_1^{-1} |h'_i|^2, \\ -(\phi_R)'_i h^2 V'_i &\leq 6 \frac{|x_i|}{|x|} \sqrt{1 - \phi_R} \cdot h^2 |V'_i| \leq 3\varepsilon_2 h^2 |V'_i|^2 (1 - \phi_R) + 3\varepsilon_2^{-1} \frac{|x_i|^2}{|x|^2} h^2, \\ -\frac{2x_i h^2 V'_i}{1+x_i^2} &\leq \varepsilon_3 h^2 |V'_i|^2 + \frac{x_i^2 h^2}{\varepsilon_3 (1+x_i^2)^2}, \end{aligned}$$

which implies by combining the above estimates subject to $\varepsilon_1 + 3\varepsilon_2 + \varepsilon_3 = 1 - a$

$$(3.2) \quad \begin{aligned} & \int U^{(i)} (a|V'_i|^2 - V''_{ii}) d\nu_i d\hat{x}_i \\ & \leq \int \frac{|h'_i|^2(1 - \phi_R)}{\varepsilon_1(1 + x_i^2)} + \frac{3|x_i|^2 h^2}{\varepsilon_2(1 + x_i^2)|x|^2} + \frac{x_i^2 h^2(1 - \phi_R)}{\varepsilon_3(1 + x_i^2)^3} d\mu. \end{aligned}$$

Step 2. Since $\frac{x_i^2}{(1+x_i^2)^3} \leq \frac{4}{27}$ for any x_i and there exists x_j with $|x_j|^2 \geq |x|^2/m$, we have

$$\sum_{i=1}^m \frac{x_i^2}{(1+x_i^2)^3} \leq \frac{4}{27}(m-1) + \frac{1}{(1+m^{-1}|x|^2)^2},$$

which implies

$$(3.3) \quad \sum_{i=1}^m \int \frac{x_i^2 h^2(1 - \phi_R)}{\varepsilon_3(1 + x_i^2)^3} d\mu \leq \left(\frac{4(m-1)}{27\varepsilon_3} + \frac{m^2}{\varepsilon_3 R^4} \right) \int_{B(0,R)^c} h^2 d\mu.$$

We also have

$$(3.4) \quad \sum_{i=1}^m \int \frac{3|x_i|^2 h^2}{\varepsilon_2(1 + x_i^2)|x|^2} d\mu \leq \frac{3}{\varepsilon_2} \int_{B(0,R)} h^2 d\mu + \frac{3m}{\varepsilon_2 R^2} \int_{B(0,R)^c} h^2 d\mu.$$

Choose R (depending on ε and $\varepsilon_{1,2,3}$) so big that $\frac{m^2}{\varepsilon_3 R^4} + \frac{3m}{\varepsilon_2 R^2} \leq \varepsilon$, then combining (3.1-3.4) gives

$$(3.5) \quad \begin{aligned} (m - \varepsilon) \int h^2 d\mu & \leq \frac{1}{\varepsilon_1} \int \sum_{i=1}^m \frac{|h'_i|^2}{1 + x_i^2} d\mu + \\ & \left(m - \varepsilon + \frac{3}{\varepsilon_2} \right) \int_{B(0,R+1)} h^2 d\mu + \left(\frac{4(m-1)}{27\varepsilon_3} + \varepsilon \right) \int h^2 d\mu. \end{aligned}$$

Step 3. We have to decide the range of ε and $\varepsilon_{1,2,3}$. First of all, fix $\varepsilon_1 < \frac{4}{27m}$, and take any ε_2 such that $\varepsilon_1 + 3\varepsilon_2 < \frac{4}{27m}$ too. It follows

$$\frac{4(m-1)}{27\varepsilon_3} = \frac{4(m-1)}{27(1-a-\varepsilon_1-3\varepsilon_2)} < m,$$

so we can take any ε such that $\frac{4(m-1)}{27\varepsilon_3} + 2\varepsilon < m$.

Now, using (3.5) yields

$$(3.6) \quad \int h^2 d\mu \leq \lambda_1 \int \sum_{i=1}^m \frac{|h'_i|^2}{1 + x_i^2} d\mu + \lambda_2 \int_{B(0,R+1)} h^2 d\mu,$$

where $\lambda_1 = [\varepsilon_1(m - 2\varepsilon - \frac{4(m-1)}{27\varepsilon_3})]^{-1}$ and $\lambda_2 = (m - \varepsilon + 3\varepsilon_2^{-1})(m - 2\varepsilon - \frac{4(m-1)}{27\varepsilon_3})^{-1}$. The proof is completed. \square

Now, we prove Proposition 1.8.

Proof. Let $\beta_n = \int |x|^{2n} d\mu$. Applying (3.6) to $h(x) = |x|^n$ yields

$$(3.7) \quad \begin{aligned} \beta_n & \leq \lambda_1 \int \sum_{i=1}^m \frac{n^2 x_i^2}{1 + x_i^2} |x|^{2n-4} d\mu + \lambda_2 \int_{B(0,R+1)} |x|^{2n} d\mu \\ & \leq \lambda_1 m n^2 \int |x|^{2n-4} d\mu + \lambda_2 (R+1)^2 \int_{B(0,R+1)} |x|^{2n-2} d\mu \\ & \leq \lambda_1 m n^2 \beta_{n-2} + \lambda_2 (R+1)^2 \beta_{n-1}, \end{aligned}$$

which implies all $\beta_n < \infty$.

For simplicity, abbreviate $\lambda'_1 = \lambda_1 m$ and $\lambda'_2 = \lambda_2(R+1)^2$. Combining the Hölder inequality with (3.7) gives

$$\beta_n = \int |x|^{n+1} |x|^{n-1} d\mu \leq \beta_{n+1}^{\frac{1}{2}} \beta_{n-1}^{\frac{1}{2}} \leq [\lambda'_1(n+1)^2 \beta_{n-1} + \lambda'_2 \beta_n]^{\frac{1}{2}} \beta_{n-1}^{\frac{1}{2}},$$

which implies

$$\beta_n \leq \frac{\lambda'_2 + \sqrt{\lambda'^2_2 + 4\lambda'_1(n+1)^2}}{2} \beta_{n-1} \leq [\lambda'_2 + \sqrt{\lambda'_1}(n+1)] \beta_{n-1}.$$

Choose any $\gamma > \sqrt{\lambda'_1}$, it follows $\lambda'_2 + \sqrt{\lambda'_1}(n+1) \leq \gamma n$ for big n , which yields a constant C such that for all n

$$\beta_n \leq C \gamma^n n!.$$

By the same argument as (2.3) for any $\delta < \gamma^{-1} < \lambda'^{-\frac{1}{2}}_1$, we have $\mu e^{\delta|x|^2} < \infty$.

Recall the constraints on all parameters (See Step 3 in the proof of Lemma 3.1), δ is allowed to be not greater than

$$\sup \left\{ \lambda'^{-\frac{1}{2}}_1 : \varepsilon_1 + 3\varepsilon_2 + \varepsilon_3 = 1 - a, \varepsilon_1 < \frac{4}{27m}, \varepsilon = \varepsilon_2 = 0 \right\},$$

which achieves $\frac{2(\sqrt{m}-\sqrt{m-1})}{3\sqrt{3m}}$. The proof is completed. \square

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